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Long-time anticipation of chaotic states in time-delay coupled ring and linear arrays

H. J. Wang, H. B. Huang, and G. X. Qi
Department of Physics, Southeast University, Nanjing 210096, China
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We study the time-delay and unidirectionally coupled ring and linear arrays of chaotic systems, and find that under certain conditions, the linear array can spatial periodically "copy" the chaotic dynamics of the ring with very long anticipation times. Numerical calculations of the Lyapunov exponents show that the delay times can result in unsynchronized chaotic waves, periodic waves, and stable states in the ring that are replicated in the linear array, but have no effect on the absolute stability of the anticipatory synchronization. Our results show that such configurations can both enhance the absolute stability of the synchronization manifolds and minimize the effects of convective instabilities.

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Chaos synchronization [1] is a universal phenomenon in nature and science, and has been intensively studied in a variety of coupled physical, chemical, biological, and social systems [2]. Many different synchronization states have been found, such as complete synchronization, phase synchronization, retarded synchronization, and generalized synchronization [2]. Of special interest is the synchronization of time-delayed chaotic systems, which is ubiquitous in nature and technology because of finite signal transmission speeds. Retarded [3] and anticipated [4] synchronization are two phenomena in time-delay systems. It has been shown that the number of positive Lyapunov exponents, and the dimensions of the attractor, increase linearly with the delay time [5]. Thus the delay times have a strong effect on the dynamics of the delay systems. This will complicate the dynamic research on delay systems. On the other hand, delay systems are good candidates for secure communication, since such systems are hyperchaotic systems.

The discovery of anticipated synchronization [4] of chaotic systems has attracted theoretical [4] and experimental [6] attention. But to our knowledge, in all the studies of anticipated synchronization, the self-time-delay feedbacks are used either in driving systems or driven systems, or both systems, i.e., the time-delayed signal of one oscillator is fed back to itself. However, it is not the case in real world, because the systems are usually driven by the time-delayed signals of other systems. Our question is: can we find the anticipated synchronization in such a system? Furthermore, the maximum stably attainable anticipation time so far was usually much shorter than the characteristic time scales of the system's dynamics. Although the anticipation time can be increased up to a multiple of the coupling delay time and can exceed the characteristic time scales of the chaotic systems by using chains of oscillators [4], the characteristic time still has a strong effect on the anticipation time and the stability of the anticipatory synchronization manifold. Can we find a more stable anticipatory synchronization manifold of chaotic systems with arbitrarily large anticipation times and minimize the convective instabilities [7] of the system?

In this Rapid Communication, we study the anticipated chaotic synchronization in unidirectionally coupled ring and linear arrays [8]. The time-delayed signal of one oscillator is used to drive the following contiguous oscillator. In our case, the characteristic time scales of the system's dynamics have no effects on the anticipation time and the absolute stability of the anticipatory synchronization manifold. As a result, we can obtain arbitrarily long anticipation times. The scheme of our coupling geometry is shown in Fig. 1, in which the linear array is driven by the circular array. All oscillators, both in the ring and in the linear array, are identical Lorenz oscillators and are connected unidirectionally with the same coupling strengths. The evolution equation for the system is

$$\dot{x}_j(t) = \sigma\big(y_j(t) - x_j(t)\big),$$

$$\dot{y}_j(t) = R[\alpha\bar{x}_j(t - T_j) + (1 - \alpha)x_j(t)] - y_j(t) - x_j(t)z_j(t),$$

$$\dot{z}_j(t) = x_j(t)y_j(t) - bz_j(t), \tag{1}$$

where the oscillators are labeled by $j=1',2',\ldots,m'$ in the ring, and $j=1,2,\ldots,N$ in the linear array. The parameters σ, R, and b are chosen in the chaotic region of the isolated Lorenz oscillator; in our case, $(\sigma,R,b)=(20,40,2.5)$. $0<\alpha<1$ is the coupling constant. The boundary conditions enter through $\bar{x}_j$, which takes the value $\bar{x}_{1'}(t-T_{1'})=x_{m'}(t-T_{1'})$ for the ring, $\bar{x}_1(t-T_1)=x_{m'}(t-T_1)$ for the linear array, and $\bar{x}_j(t-T_j)=x_{j-1}(t-T_j)$ for $j(j')\neq 1$. We should note that, in our scheme, the jth [or j'th] oscillator is driven by the time-delayed signal $x_{j-1}(t-T_j)$ [or $x_{j'-1}(t-T_{j'})$] of the (j-1)th [or

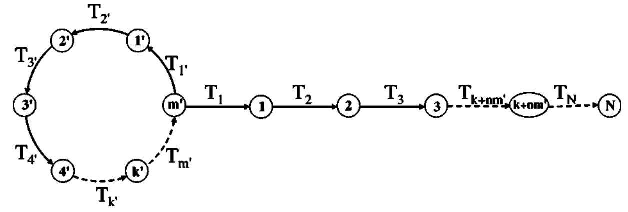

FIG. 1. Geometry of the coupled ring and linear arrays of Lorenz systems.

$(j'$-1)th] oscillator. There are no self-delay feedbacks of individual oscillators. If T_j $[T_{j'}]=0$, then any m' neighboring oscillators in the linear array respond to the ring and exhibit the same dynamic behaviors as that in the ring in the long time limit, i.e., the noncontiguous oscillators k' and $k+nm'$ $(n=0,1,2,\ldots;k=k')$ become chaotic synchronized with the spatial period m' in the linear array [8]. While if $T_j(T_{j'})$ $\neq 0$, the $k+nm'$ oscillator in the linear array will anticipate the chaotic state of the oscillator k' in the ring with anticipation time

$$\tau_k = T_{k'-(k+nm')} = \sum_{i=i'=1}^{k=k'} (T_{i'} - T_i) + n\sum_{i'=1}^{m'} T_{i'} - \sum_{i=1}^{nm'} T_{k+i} = \tau_{k-1} + T_{k'} - T_{k+nm'}. \qquad (2)$$

Equation (2) can easily be obtained from the stability analysis of the anticipatory synchronization manifolds (see our later discussion). The values of T_i and $T_{i'}$ $[0<T_i(T_{i'})<\infty]$ have no effects on the absolute stabilities of the anticipatory synchronization manifolds in our coupling configuration, so we can obtain an arbitrarily large anticipation time $\tau_k(0<\tau_k<\infty)$ by letting either $T_{i'}$'s$\gg T_i$'s or n be large (long chain). On the other hand, the lag chaotic synchronization also exists for $T_{i'}$'s$<T_i$'s. In fact, we can obtain an arbitrary $\tau_k(-\infty<\tau_k<\infty)$ in this system. Before discussing the stability of the anticipatory synchronization, we should note that different $T_{i'}$ can cause different dynamic states in the ring. As an example, taking $m'=3$, and $T_{1'}=T_{2'}=T_{3'}=T$, we numerically simulate the dynamics in the ring by calculating the largest Lyapunov exponents for different T and α (see Fig. 2). The white, gray, and black regions in Fig. 2(a) correspond to chaotic [see Figs. 2(b) and 2(f)], periodic [see Figs. 2(d) and 2(c)], and stable [see Fig. 2(c)] states, respectively. Only chaotic states exist in the ring when $T\gtrsim 1.5$. All chaotic motions for $T\gtrsim 0.03$, which consist of spirals around the zero fixed point as $t\to\infty$, are unsynchronous. These states are replicated in the linear array with anticipation time τ_k if the coupling constant α is larger than the threshold α_c (≈ 0.1 in our case). From the above discussions we see that the time $\tau_k=T_{k'-(k+nm')}$ in the linear array can be positive, zero, or negative, that is to say, the oscillator $k+nm'$ in the linear array can have anticipated synchronous, complete synchronous, and retarded synchronous states, compared with the chaotic states of the oscillator k' in the ring. Figure 3 shows the wave forms of the chaotic states of the oscillator $1'$ (dashed traces), and $1+3n$ $(n=0,1,2)$ (solid traces) for the following parameters $m'=3$, $T_{1'}=5$, $T_{2'}=10$, $T_{3'}=15$, $T_1=T_2=T_3=10$, $T_4=5$, $T_5=T_6=10$, $T_7=5$, and $\alpha=0.3$. Equation (2) gives $T_{1'-1}=-5$ (retarded synchronization), $T_{1'-4}=0$ (complete synchronization), and $T_{1'-7}=5$ (anticipated synchronization), which has the same results with the numerical simulations in Fig. 3 (numerical accuracy is 10^{-12}).

The degree of the various synchronizations discussed above and the time shifts can be quantified by calculating the correlation function [9]

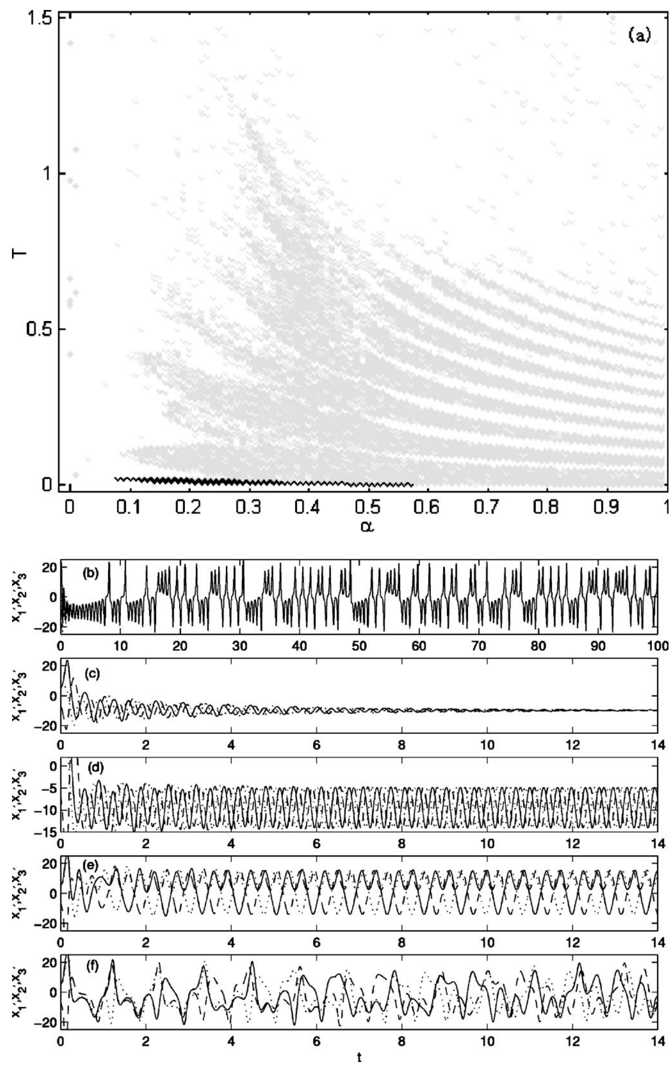

FIG. 2. (a) The largest Lyapunov exponent λ^1 in α-T parameter space. The white regimes denote where $\lambda^1>0$, the gray regimes denote where $\lambda^1=0$, and the black regimes denote where $\lambda^1<0$. (b)–(f) The numerically simulated time series $x_{1'}$ (solid line), $x_{2'}$ (dashed line), $x_{3'}$ (dotted line) for the following different T in the ring. We let $\alpha=0.3$, $T_{1'}=T_{2'}=T_{3'}=T$, $T=0.005$ in (b), $T=0.015$ in (c), $T=0.02$ in (d), $T=0.2$ in (e), $T=2$ in (f), and $(\sigma,R,b)=(20,40,2.5)$ for (a)–(f).

$$S^2(\tau) = \frac{\langle [x_{k'}(t+\tau) - x_{k+nm'}(t)]^2\rangle}{[\langle x_{k'}^2(t)\rangle\langle x_{k+nm'}^2(t)\rangle]^{1/2}}. \qquad (3)$$

Figures 4(a)–4(c) show $S(\tau)$ obtained from the corresponding traces in Fig. 3. The minimums are shown at $\tau=-5$, 0, and 5, which indicate the retarded, complete, and anticipated synchronization, respectively. There are also additional minimums at $\tau\pm\Sigma_{i'=1}^{3}T_{i'}=\tau\pm 30l$ $(l=1,2,\ldots)$, which arise from time correlations of the chaotic waves in the ring.

To study the absolute stability of the anticipatory synchronization manifold $\boldsymbol{x}_{k'}=\boldsymbol{x}_{k+nm',\tau_k}=\boldsymbol{x}_{k+nm'}(t-\tau_k)$ with $\tau_k=T_{k'-(k+nm')}$, and $\boldsymbol{x}_j=(x_j,y_j,z_j)^T$, we numerically calculate the

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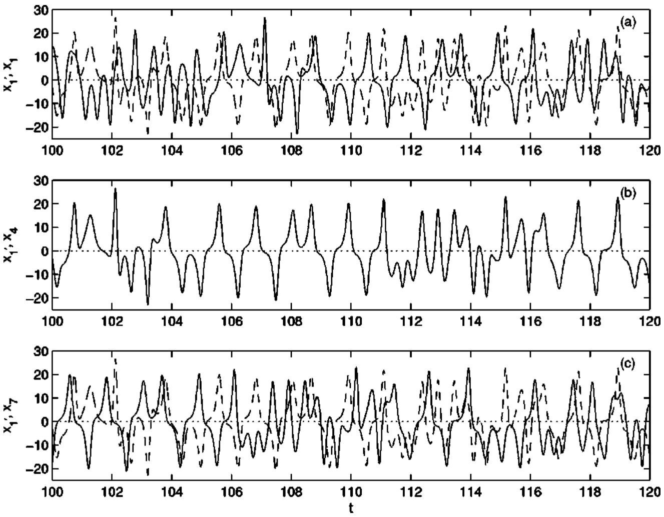

FIG. 3. Numerically simulated time series $x_{1'}$ (dashed line), and x_1,x_4,x_7 (solid line). The dotted lines indicate the value of $x_i(t-\tau_k)-x_{1'}(t)$ $(i=1,4,7)$. (a) retarded synchronization, (b) complete synchronization, (c) anticipated synchronization. The parameters are $m'=3$, $T_{1'}=5$, $T_{2'}=10$, $T_{3'}=15$, $T_1=T_2=T_3=10$, $T_4=5$, $T_5=T_6=10$, $T_7=5$, $(\sigma,R,b)=(20,40,2.5)$, and $\alpha=0.3$.

maximum transversal Lyapunov exponent of the transversal system $\mathbf{\Delta}^{(\tau_k)}=\mathbf{x}_{k'}-\mathbf{x}_{k+nm',\tau_k}=(\Delta_x^{(\tau_k)},\Delta_y^{(\tau_k)},\Delta_z^{(\tau_k)})^T$. From Eq. (1) we have

$$\dot{\mathbf{\Delta}}^{(\tau_k)}=\begin{pmatrix} -\sigma & \sigma & 0 \\ R(1-\alpha)-z_{k+nm',\tau_k} & -1 & -x_{k'} \\ y_{k'} & x_{k+nm',\tau_k} & -b \end{pmatrix}\mathbf{\Delta}^{(\tau_k)} + R\alpha\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{\Delta}^{(\tau_{k-1})}, \qquad (4)$$

where $\mathbf{\Delta}^{(\tau_{k-1})}=\mathbf{x}_{k'-1}(t-T_{k'})-\mathbf{x}_{k-1+nm'}(t-T_{k+nm'}-\tau_k)$ $:=\mathbf{x}_{k'-1}(t)-\mathbf{x}_{k-1+nm'}(t-\tau_{k-1})$. It is obvious that $\mathbf{\Delta}^{(\tau_k)}=0$ and $\mathbf{\Delta}^{(\tau_{k-1})}=0$ are fixed points of this system, and therefore $\mathbf{x}_{k'}(t)=\mathbf{x}_{k+nm'}(t-\tau_k)$, $\mathbf{x}_{k'-1}(t)=\mathbf{x}_{k-1+nm'}(t-\tau_{k-1}),\ldots,$ and $\mathbf{x}_{k'-m'}(t)=\mathbf{x}_{k+(n-1)m'}(t-\tau_{k-m'})$ are anticipatory synchronization manifolds. These manifolds can be shown to be stable by calculating the maximum transverse Lyapunov exponent based on Eq. (1) and Eq. (4). The results for $m'=3$ show that the anticipatory manifold $\mathbf{x}_{1'}(t)=\mathbf{x}_1(t-\tau_1)$ is stable for arbitrary τ_1 if $\alpha \gtrsim \alpha_c$ ($\alpha_c\approx 0.1$ in our case). The above discussions are also true for $\tau_k<0$ (retarded synchronization).

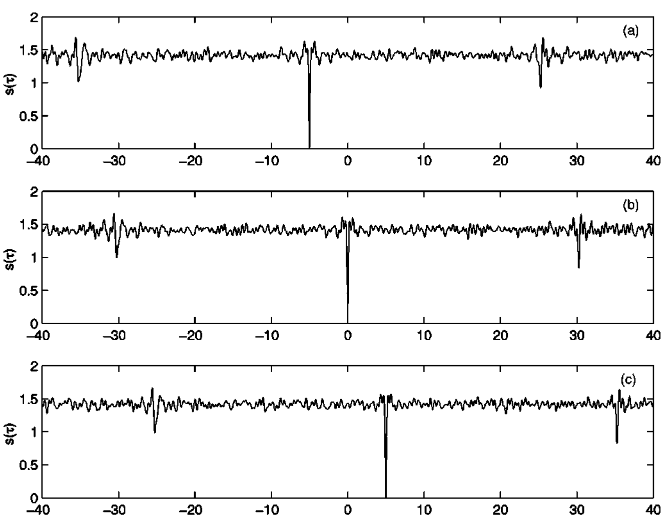

FIG. 4. Correlation function $S(\tau)$ calculated for the corresponding traces in Figs. 3(a)–3(c).

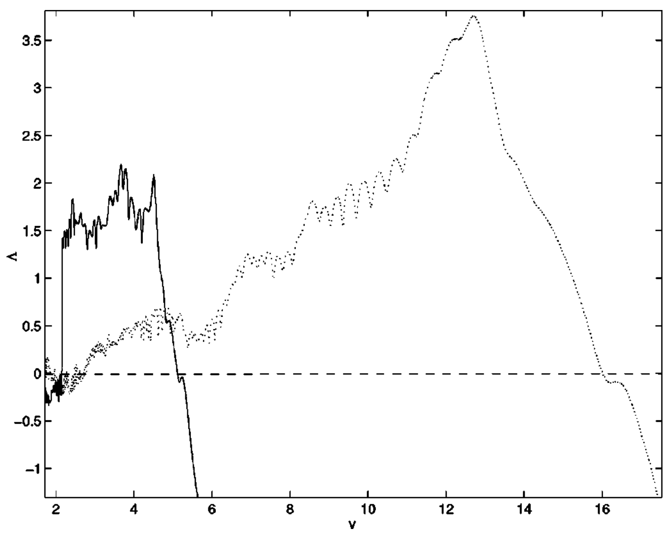

FIG. 5. Convective Lyapunov exponent $\Lambda(v)$ vs propagation velocity $v(=j/t)$ for both the ring-linear array system (solid line) and the chain system (dotted line). The δ-like perturbation $\eta=10^{-10}$ was added to the first Lorenz oscillators of both systems. The parameters $(\sigma,R,b;\alpha)=(20,40,2.5;0.5)$ for both coupling schemes, $\tau=0.001$, $T'=0.002$, and $T=0.001$.

But the absolute stability of the anticipatory synchronization manifolds discussed above are sensitive to the perturbation or noise, and which can be demonstrated by calculating the convective Lyapunov exponent $\Lambda(v)=\lim_{t\to\infty}(1/t)\ln[\delta(j=vt,t)]/[\delta(0,0)]$ [7]. Figure 5 shows the numerical results of the convective Lyapunov exponents for both the ring-linear array system, and, as a comparison, the chain system considered in Refs. [4,7]. In the chain [4,7], we also choose the y coupling scheme as in Eq. (1), but the coupling method is similar to that in Refs. [4,7]: $\dot{y}_j(t)=Rx_j(t)-y_j(t)-x_j(t)z_j(t)+\alpha R[x_{j-1}(t)-x_j(t-\tau)]$. The δ-like perturbation $\delta(0,0)$, which propagates with velocity $v(=j/t)$ was added to the first Lorenz oscillators in both schemes. From Fig. 5 we can see that the perturbation propagating with a velocity v in between the two zeros of $\Lambda(v)$ (approximately equal to 2 and 5 for the ring-linear array system, and 2 and 16 for the chain system) are amplified from $t\geqslant 0$, while for Rössler systems, our numerical calculation shows that the amplification takes place at $t\gtrsim 2$ for both schemes. Figure 5 also shows that the Δv [in which $\Lambda(v)>0$] of the chain system is larger than that of the ring-linear array system, i.e., the chain system is more unstable than the ring-linear array system. The similar situation occurs for Rössler systems.

In conclusion, we have constructed a unidirectionally and time-delay coupled ring and linear array system, which can exhibit spatial periodic (period m') anticipated, complete, and retarded chaos synchronization with arbitrarily large anticipation or retarded time τ_k. Numerical simulations show that the absolute stabilities of anticipatory and retardatory synchronization manifolds are the same as those for identical synchronization. The spectra of the convective Lyapunov exponent shows that the ring-linear array scheme can minimize the effects of convective instabilities compared with the chain scheme. But the stabilities of the synchronization manifolds are very sensitive to the noise or perturbation in both the ring-linear array system and the chain system [4,7]. So the absolute stability is only a necessary condition for the synchronization manifolds [7]. Our system has no self-delay feedback in individual oscillator, and could have potential applications in secure communication and neural processes.

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